Basics of wetting: The Young-Laplace equation

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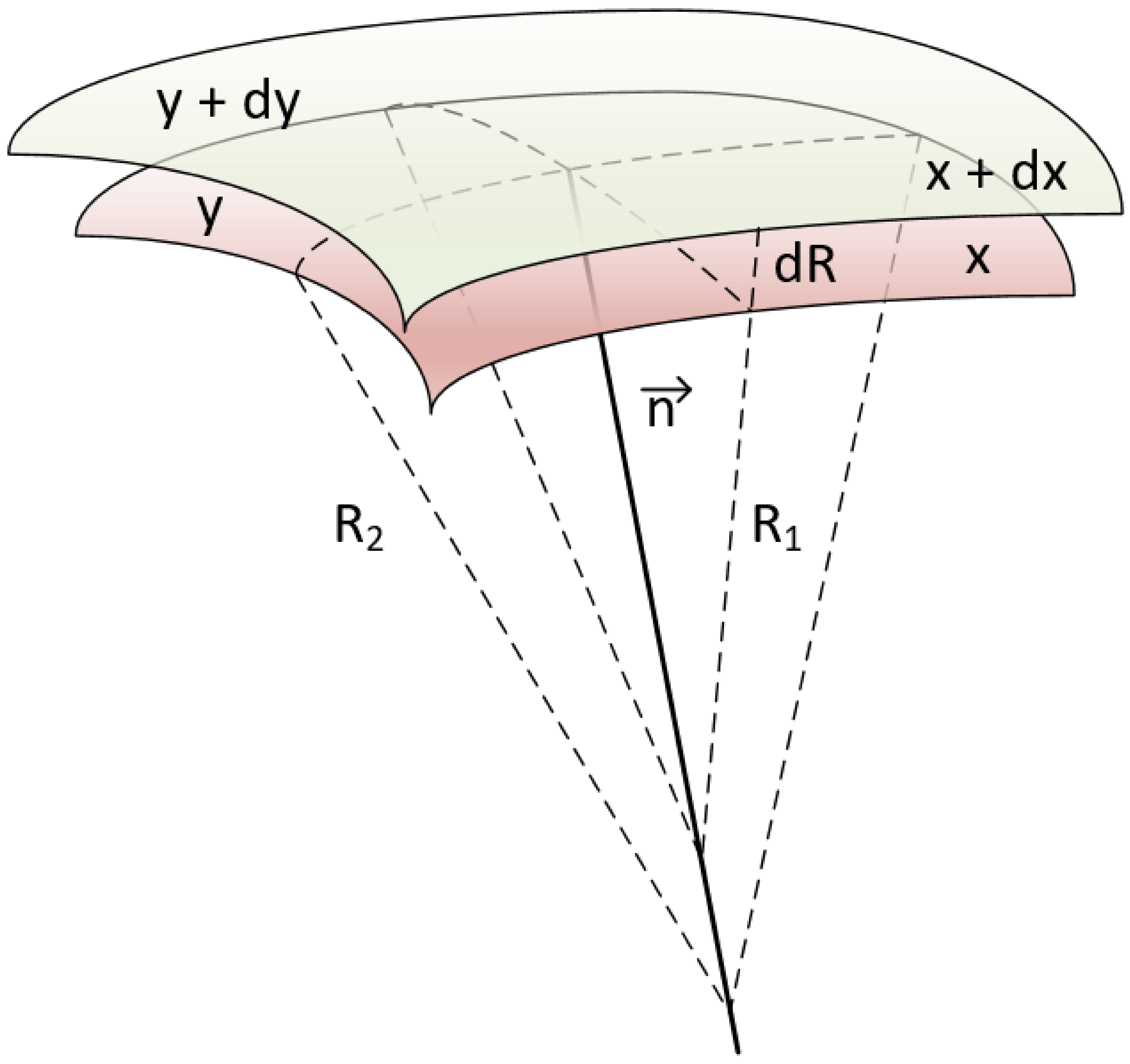
Introduction

The shape of liquid drop is governed by what is known as the Young-LaPlace equation. It was derived more or less simultaneously by Thomas Young (1804) and Simon Pierre de Laplace(1805). A short derivation of this equation is presented here.



Consider a small section of a curved surface with carthesian dimensions x and y. On the normal n two arcs can be constructed that support the lengths of x and y:

$$x = \alpha R_1 \text{ and } y = \beta R_2 \quad (1)$$

When the radii are enlarged by a small amount dR the work dW performed equals:

$$dW = \gamma\,(x\,dy + y\,dx) \quad (2)$$

Where γ is the surface energy. The work is done against a pressure. According to thermodynamics this work can also be written as:

$$dW = P\,dV = P\,x\,y\,dR \quad (3)$$

So we obtain for P:

$$P = \frac{1}{x\,y\,dR}\gamma\,(x\,dy + y\,dx) \quad (4)$$

From the first equation it can simply be shown that:

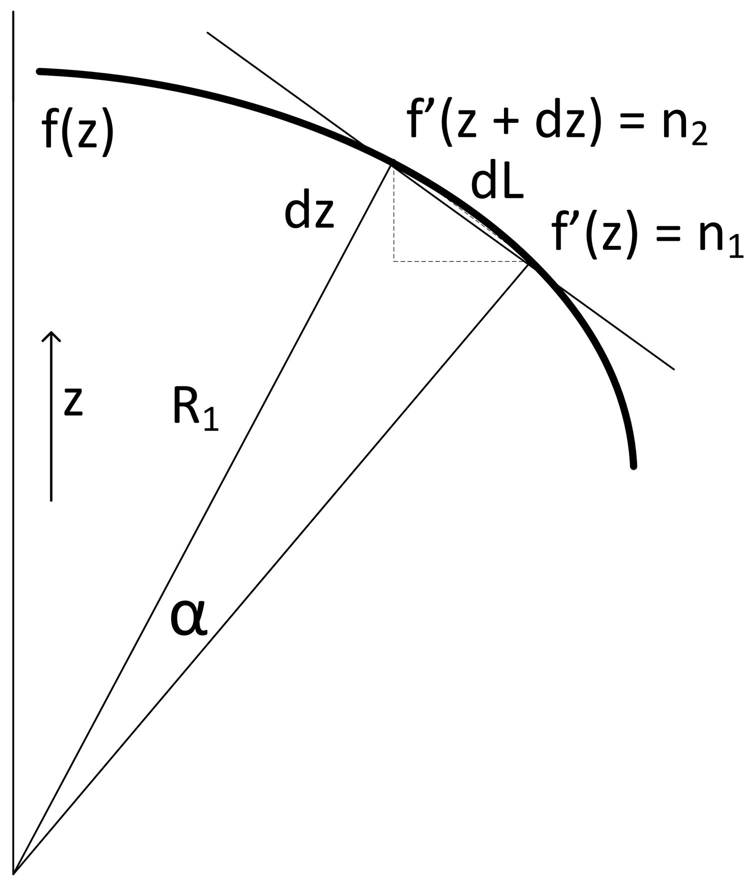
$$dx = x\,\frac{dR}{R_1} \text{ and } dy = y\,\frac{dR}{R_2} \quad (5)$$

Insertion of the results from (5) into eq (4 yields the Young-LaPlace equation:

$$P = \gamma\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

The radii of curvature

An expression for R1 for the cylindrical symmetrical can be deduced as follows:



Consider a contour f(z) with derivative f'(z) equaling n_1. The length of the segment dL over distance dz equals:

$$dL = dz\,(1 + f'(z)^2)^{1/2}$$

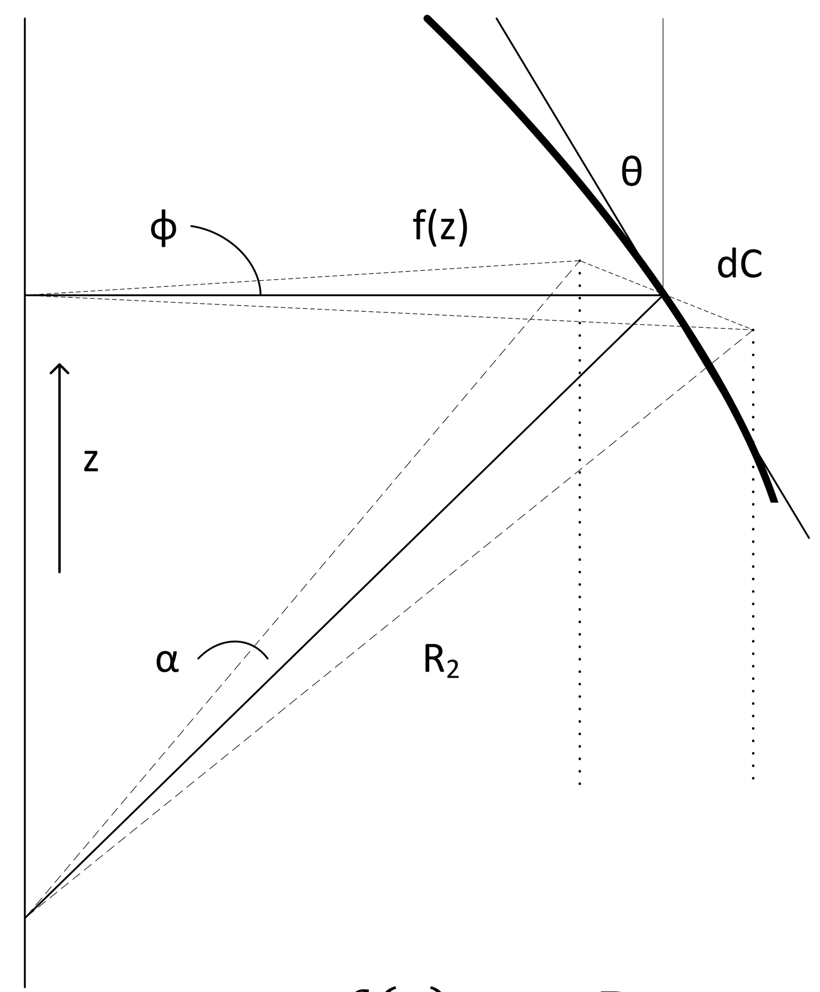
This is equated to $\alpha.R_1$. The angle α is derived from the difference in derivatives in z and z+dz:

$$\alpha \cong \tan(\alpha) = \frac{n_1 - n_2}{1 + n_1 n_2} = -\,dz\,\frac{f''(z)}{1 + f'(z)^2}$$

Upon completion we obtain for R_1:

$$R_1 = -\frac{(1 + f'(z)^2)^{3/2}}{f''(z)}$$

The second radius R_2 is derived as follows: take a small segment from the circumference of the profile. From the length of this segment dC we can see that:



$$\varphi\,f(z) = \alpha\,R_2$$

From the figure we learn that:

$$\alpha = \varphi\cos(\theta)$$

From which it can readily be deduced that:

$$R_2 = f(z)(1 + f'(z)^2)^{1/2}$$

As a final result we obtain or the Young-Laplace equation:

$$P = \gamma\left(-\frac{f''(z)}{(1 + f'(z)^2)^{3/2}} + \frac{1}{f(z)(1 + f'(z)^2)^{1/2}}\right)$$